

Sl No of QP : 5713  
Unique Paper Code : 42354401  
Name of the Paper : Real Analysis  
Name of the Course : B.Sc. (Prog) Physical Sciences/Mathematical Sciences  
Semester : IV  
Duration : 3 Hours

Maximum Marks: 75

(Write your Roll No. on the top immediately on receipt of this question paper.)

All questions are compulsory.

Attempt any two parts from each question.

1. (a) Define a countable set. Show that the set  $\mathbb{Q}$  of rational numbers is countable.

(b) Define absolute value of a real number ' $x$ '.

Find all  $x \in \mathbb{R}$  that satisfy the following inequalities:

(i)  $4 < |x + 2| + |x - 1| < 5$

(ii)  $|2x - 1| \leq x + 1$

(c) (i) Define supremum of a non-empty bounded subset  $S$  of  $\mathbb{R}$ .

(ii) Show that a real number  $u$  is the supremum of a non-empty subset  $S$  of  $\mathbb{R}$  if and only if it satisfies the following conditions:

(1)  $s \leq u$  for all  $s \in S$ .

(2) For each positive real number  $\varepsilon$ , there exists  $s_\varepsilon \in S$  such that  $u - \varepsilon < s_\varepsilon$ . (6,6)

2. (a) State the Archimedean Property of real numbers. Show that if  $x \in \mathbb{R}$ , then there exists a unique  $n \in \mathbb{Z}$  such that  $n - 1 \leq x < n$ .

(b) Define the convergence of a sequence  $(x_n)$  of real numbers. Using the definition, evaluate the following limits:

(i)  $\lim_{n \rightarrow \infty} \sqrt{n+1} - \sqrt{n}$

(ii)  $\lim_{n \rightarrow \infty} \left( \frac{(-1)^n n}{n^2 + 1} \right)$

(c) Let  $(x_n)$  be a sequence of real numbers that converges to  $x$  and suppose that  $x_n \geq 0$ ,

$\forall n \in \mathbb{N}$ . Show that the sequence  $(\sqrt{x_n})$  converges to  $\sqrt{x}$ .

(6,6)

3. (a) Prove that every monotonically decreasing and bounded below sequence of real numbers converges.

(b) Show that the sequence  $(x_n)$  defined by

$x_1 = 1; x_{n+1} = \frac{1}{4}(2x_n + 3), \forall n \geq 1$  is convergent. Also, find  $\lim_{n \rightarrow \infty} x_n$ .

(c) State Cauchy's Convergence Criterion for sequences of real numbers. Show directly from the definition that the following sequence is a Cauchy sequence:

$$\left( 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} \right).$$

(6.5, 6.5)

4. (a) State and prove Comparison test for positive term series. Hence, show that the following series converges:

$$1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \dots$$

- (b) Suppose that  $(x_n)$  is a sequence of non-negative real numbers. Prove that the series  $\sum x_n$  converges if and only if the sequence  $S = (s_k)$  of partial sums is bounded.

- (c) (1) State (without proof) D'Alembert's ratio test for an infinite series.

(2) Test for convergence the series:

(i)  $\frac{1}{3} + \frac{1.2}{3.5} + \frac{1.2.3}{3.5.7} + \dots$

(ii)  $\frac{1}{\log 2} + \frac{1}{(\log 3)^2} + \frac{1}{(\log 4)^3} + \dots$

(6.5, 6.5)

5. (a) (i) Define an absolutely convergent series. Is every convergent series absolutely convergent? Justify your answer.

(ii) Test for convergence the series:

(1)  $1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{2^4} - \dots$

(2)  $\sum_{n=1}^{\infty} (-1)^n \cdot e^{-n}$

- (b) Show that if  $a > 0$ , then the sequence  $\left( \frac{nx}{1+n^2x^2} \right)$  converges uniformly on the interval  $[a, \infty)$  but not uniformly on the interval  $[0, \infty)$ .

- (c) State Weierstrass M-Test for uniform convergence of series. Hence, show that

$$\sum \frac{1}{x^2 + n^2}, \quad \forall x \in \mathbf{R}$$

is uniformly convergent.

(6.5, 6.5)

6. (a) Find the radius of convergence and exact interval of convergence of the power series

$$\sum \frac{n+1}{(n+2)(n+3)} x^n.$$

- (b) Show that the function  $f(x) = x^2$  defined on the interval  $[0, b]$ , where  $b > 0$  is Riemann integrable.

- (c) Show that every continuous function defined on  $[a, b]$  is Riemann integrable.

(6.6)